PERIOD SOLUTIONS OF QUASILINEAR NON-SELF-CONTAINED SYSTEMS WITH ONE DEGREE OF FREEDOM IN PARTICULAR CASES

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Let us consider the non-self-contained system of the form

$$x^{\cdot\cdot} + m^2 x = f(t) + \mu F(t, x, x^{\cdot}, \mu) \tag{0.1}$$

We shall assume that $F(t, x, x^{\cdot}, \mu)$ is an analytical function of x, $x^{\cdot}$ and the small parameter μ in some domain of variation x and $x^{\cdot}$ for $0 \leqslant \mu < \mu_0$. Furthermore, F and f are continuous periodic functions of the time t having a period equal to 2π. The quantity m can be an integer or zero. In the first case, the Fourier series of the function $f(t)$ does not have any m-th order harmonics, and in the other case it does not have a constant term.

In the papers [1, 2 and 3] it is shown how periodic solutions of the system (0.1) can be obtained in the vicinity of the fundamental resonance for simple and repeated roots of the amplitude equations for $m \neq 0$. It was assumed then, that at least one of the elements of the functional determinant of the system of equations yielding β and γ is different from zero. In the present paper consideration is given to the case in which all the elements of the functional determinant are equal to zero, or when $m = 0$.

1. All the elements of the functional determinant are equal to zero.

The generating function for $\mu = 0$ has the general periodic solution

$$x_0(t) = \varphi(t) + A_0 \cos mt + B_0 m^{-1} \sin mt \tag{1.1}$$

depending on two arbitrary constants A_0 and B_0.

We search periodic solutions of Eq. (0.1) using Poincaré's method. Let us take for initial conditions

$$x(0) = \varphi(0) + A_0 + \beta, \quad x^{\cdot}(0) = \dot{\varphi}(0) + B_0 + \gamma \tag{1.2}$$

Here β and γ are functions of the parameter μ vanishing for $\mu = 0$.

The solution of Eq. (0.1) is an analytic function of $A_0 + \beta$, $B_0 + \gamma$ and μ. Let us represent it in the form

$$x(t) = \varphi(t) + (A_0 + \beta)\cos mt + \frac{B_0 + \gamma}{m} \sin mt + $$
$$+ \sum_{n=1}^{\infty} \left[C_n(t) + \frac{\partial C_n(t)}{\partial A_0}\beta + \frac{\partial C_n(t)}{\partial B_0}\gamma + \frac{1}{2}\frac{\partial^2 C_n(t)}{\partial A_0^2}\beta^2 + \ldots\right]\mu^n \tag{1.3}$$

The functions $C_n(t)$ are determined by the relations

$$C_n(t) = \frac{1}{m}\int_0^t H_n(t')\sin m(t-t')\,dt', \qquad H_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}F}{d\mu^{n-1}}\right)_{\beta=\gamma=\mu=0} \quad (1.4)$$

The values of the functions $H_n(t)$ for $n = 1, 2, 3$ are derived in [1]. From the periodicity conditions we may derive the equations for the fundamental amplitudes A_0 and B_0

$$C_1(2\pi, A_0, B_0) = 0, \quad C_1^{\cdot}\ (2\pi, A_0, B_0) = 0 \quad (1.5)$$

and also the equations for determining the parameters β and γ as implicit functions of μ.

Grouping the terms entering these equations in the form of homogeneous polynomials, we get

$$\frac{\partial C_1}{\partial A_0}\beta + \frac{\partial C_1}{\partial B_0}\gamma + C_2\mu + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}\beta^2 + \frac{\partial^2 C_1}{\partial A_0 \partial B_0}\beta\gamma + \frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2}\gamma^2 +$$
$$+ \frac{\partial C_2}{\partial A_0}\beta\mu + \frac{\partial C_2}{\partial B_0}\gamma\mu + C_3\mu^2 + \frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}\beta^3 + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^2 \partial B_0}\beta^2\gamma +$$
$$+ \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0 \partial B_0^2}\beta\gamma^2 + \frac{1}{6}\frac{\partial^3 C_1}{\partial B_0^3}\gamma^3 + \frac{1}{2}\frac{\partial^2 C_2}{\partial A_0^2}\beta^2\mu + \frac{\partial^2 C_2}{\partial A_0 \partial B_0}\beta\gamma\mu + \cdots = 0 \quad (1.6)$$

and an analogous equation in which all the C_n are replaced by $C_n^{\cdot}$. In these formulas all the functions $C_n(t)$ and $C_n^{\cdot}(t)$ and their derivatives with respect to A_0 and B_0 are taken for $t = 2\pi$.

A necessary and sufficient condition for the roots of the amplitude equations to be repeated is that the functional determinant

$$D_1 = \frac{\partial C_1}{\partial A_0}\frac{\partial C_1^{\cdot}}{\partial B_0} - \frac{\partial C_1}{\partial B_0}\frac{\partial C_1^{\cdot}}{\partial A_0} = 0 \quad (1.7)$$

be equal to zero.

In the general case it is possible to determine the number of times a root is repeated by considering whether the determinants $D_2, D_3 \ldots$ etc, are equal to zero [3]. In this case

$$\frac{\partial C_1}{\partial A_0} = \frac{\partial C_1}{\partial B_0} = \frac{\partial C_1^{\cdot}}{\partial A_0} = \frac{\partial C_1^{\cdot}}{\partial B_0} = 0 \quad (1.8)$$

on the basis of the roots of Eqs. (1.5). Thus all the determinants D_n ($n = 2, 3, \ldots$) are equal to zero and do not enter the solution of the given problem.

We shall assume, as was done in [3], that each of Eqs. (1.5) determines a curve on the plane of the amplitudes A_0B_0. The points of intersection of the curves represent the roots of these equations.

Let us differentiate twice Eqs. (1.5) with respect to A_0, assuming that B_0 is a function of A_0. Taking (1.8) into consideration, we have

$$\frac{\partial^2 C_1}{\partial A_0^2} + 2\frac{\partial^2 C_1}{\partial A_0 \partial B_0}\frac{dB_0}{dA_0} + \frac{\partial^2 C_1}{\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2 = 0, \quad \frac{\partial^2 C_1^{\cdot}}{\partial A_0^2} + 2\frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0}\frac{dB_0}{dA_0} + \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2 = 0 \quad (1.9)$$

At the points of double intersection of the curves the tangents to those curves must coincide. Thus the resultant of the quadratic Eqs. (1.9) with respect to dB_0/dA_0 must be equal to zero. We have

$$D_1^* = 4\left(\frac{\partial^2 C_1}{\partial A_0^2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0} - \frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\frac{\partial^2 C_1}{\partial A_0 \partial B_0}\right)\left(\frac{\partial^2 C_1}{\partial B_0^2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0} - \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\frac{\partial^2 C_1}{\partial A_0 \partial B_0}\right) +$$

$$+\left(\frac{\partial^2 C_1}{\partial A_0^2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}-\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\frac{\partial^2 C_1}{\partial B_0^2}\right)^2=0 \tag{1.10}$$

Let us differentiate (1.5) with respect to A_0 a third time. We get

$$\Phi_3(C_1)+\Phi_2(C_1)\frac{d^2B_0}{dA_0^2}=0, \qquad \Phi_3(C_1^{\cdot})+\Phi_2(C_1^{\cdot})\frac{d^2B_0}{dA_0^2}=0 \tag{1.11}$$

In these formulas we have denoted

$$\Phi_2(C_1)=3\left(\frac{\partial^2 C_1}{\partial A_0\partial B_0}+\frac{\partial^2 C_1}{\partial B_0^2}\frac{dB_0}{dA_0}\right) \tag{1.12}$$

$$\Phi_3(C_1)=\frac{\partial^3 C_1}{\partial A_0^3}+3\frac{\partial^3 C_1}{\partial A_0^2\partial B_0}\frac{dB_0}{dA_0}+3\frac{\partial^3 C_1}{\partial A_0\partial B_0^2}\left(\frac{dB_0}{dA_0}\right)^2+\frac{\partial^3 C_1}{\partial B_0^3}\left(\frac{dB_0}{dA_0}\right)^3$$

and similarly for C_1^*. Let us form the determinant of the system (1.11)

$$D_2^*=\Phi_3(C_1)\Phi_2(C_1^{\cdot})-\Phi_3(C_1^{\cdot})\Phi_2(C_1) \tag{1.13}$$

If $D_2^*=0$, the Eqs. (1.5) have at least a root repeated three times. Consequently, $D_2^*\neq 0$ is a necessary and sufficient condition for a double root. We shall stop at this point the analysis of the order of the roots.

Let us examine in detail the case of double roots. Let us assume that β and γ can be expanded in power series of $\mu^{1/2}$. From Expansion (1.6) and analogously to it we find the equations for the coefficients $A_{1/2}$ and $B_{1/2}$.

$$\begin{aligned}\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2}A_{1/2}^2+\frac{\partial^2 C_1}{\partial A_0\partial B_0}A_{1/2}B_{1/2}+\frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2}B_{1/2}^2+C_2=0\\ \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}A_{1/2}^2+\frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}A_{1/2}B_{1/2}+\frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}B_{1/2}^2+C_2^{\cdot}=0\end{aligned} \tag{1.14}$$

If $C_2=C_2^*=0$, the coefficients $A_{1/2}$ and $B_{1/2}$ become equal to zero. We shall assume that one of the quantities C_2 or C_2^* is not equal to zero; then $A_{1/2}\neq 0$ and $B_{1/2}\neq 0$. Using the relations (1.10) we may transform the system (1.14) into the form:

$$K_1A_{1/2}^2=K_2, \quad L_1B_{1/2}^2=L_2 \tag{1.15}$$

The coefficients K_1 and K_2 have the values

$$K_1=\left(\frac{\partial^2 C_1}{\partial A_0^2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}-\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\frac{\partial^2 C_1}{\partial B_0^2}\right)\left(\frac{\partial^2 C_1}{\partial B_0^2}C_2^{\cdot}-\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}C_2\right)+$$

$$+2\left(\frac{\partial^2 C_1}{\partial B_0^2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}-\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}\frac{\partial^2 C_1}{\partial A_0\partial B_0}\right)\left(\frac{\partial^2 C_1}{\partial A_0\partial B_0}C_2^{\cdot}-\frac{\partial^2 C_1^{\cdot}}{\partial A_0\partial B_0}C_2\right)$$

$$K_2=\left(\frac{\partial^2 C_1}{\partial B_0^2}C_2^{\cdot}-\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2}C_2\right)^2$$

and the coefficients L_1 and L_2 can be obtained from the coefficients K_1 and K_2 by replacing the differentiation with respect to A_0 by a differentiation with respect to B_0 and vice versa.

The Eqs. (1.15) have either two real roots or none. The equations for the ensuing coefficients are linear. Consequently, the Eqs. (0.1) will have either two real solutions which can be expanded in powers of $\mu^{1/2}$ or none. For the coefficients A_1 and B_1 we get

$$\frac{\partial^2 C_1}{\partial A_0^2}A_{1/2}A_1+\frac{\partial^2 C_1}{\partial A_0\partial B_0}(A_{1/2}B_1+B_{1/2}A_1)+\frac{\partial^2 C_1}{\partial B_0^2}B_{1/2}B_1+\frac{1}{6}\frac{\partial^3 C_1}{\partial A_0^3}A_{1/2}^3+ \tag{1.16}$$

$$+\frac{1}{2}\frac{\partial^3 C_1}{\partial A_0^2 \partial B_0} A_{1/2}^2 B_{1/2} + \frac{1}{2}\frac{\partial^3 C_1}{\partial A_0 \partial B_0^2} A_{1/2} B_{1/2}^2 + \frac{1}{6}\frac{\partial^3 C_1}{\partial B_0^3} B_{1/2}^3 + \frac{\partial C_2}{\partial A_0} A_{1/2} + \frac{\partial C_2}{\partial B_0} B_{1/2} = 0$$

and an analogous equation obtained by replacing all the C_n by $C_n^{\cdot}$. It can be shown that the determinant of these equations is not equal to zero.

If $A_{1/2} = B_{1/2} = 0$, the coefficients A_1 and B_1 are determined from the system

$$\frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^2} A_1^2 + \frac{\partial^2 C_1}{\partial A_0 \partial B_0} A_1 B_1 + \frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2} B_1^2 + \frac{\partial C_2}{\partial A_0} A_1 + \frac{\partial C_2}{\partial B_0} B_1 + C_3 = 0 \quad (1.17)$$

$$\frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2} A_1^2 + \frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0} A_1 B_1 + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} B_1^2 + \frac{\partial C_2^{\cdot}}{\partial A_0} A_1 + \frac{\partial C_2^{\cdot}}{\partial B_0} B_1 + C_3^{\cdot} = 0 \quad \text{etc.}$$

Expressions for the determination of the coefficients of the expansions of the solution of Eq. (0.1) in series of integer or fractional powers of the parameter μ are derived in [1 and 3]. For a practical determination of those coefficients, it is sometimes easier to find them by means of a successive integration of the equations which determine them.

Let us consider the example

$$x^{\cdot\cdot} + x = \mu\,(ax^3 + bx^{\cdot 3}) + \mu^2\,(\nu \cos t + \lambda \sin t) \quad (1.18)$$

We have the generating function

$$x_0(t) = A_0 \cos t + B_0 \sin t \quad (1.19)$$

Let us construct the amplitude Eqs. (1.20)

$$C_1(2\pi) = {}^3/_4\,\pi\,(bA_0 - aB_0)(A_0^2 + B_0^2), \qquad C_1^{\cdot}(2\pi) = {}^3/_4\,\pi\,(aA_0 + bB_0)(A_0^2 + B_0^2)$$

The roots of these Eqs. are

$$A_0 = B_0 = 0 \quad (1.21)$$

All the first and second derivatives of C_1 and $C_1^{\cdot}$ with respect to A_0 and B_0 are equal to zero. Thus we get $D_1^* = D_2^* = 0$. Let us compute the third derivatives of the indicated quantities and substitute them into Eqs. (1.11). From these relations it can be seen that two cubic equations with respect to dB_0 / dA_0 have a common factor equal to $(dB_0 / dA_0)^2 + 1$. Consequently the roots of (1.21) are repeated three times.

Let us seek β and γ in the form of series in the powers of $\mu^{1/3}$ We substitute the values of the third derivatives of C_1 and $C_1^{\cdot}$, and also the quantities $C_2 = -\pi\lambda$ and $C_2^{\cdot} = \pi\nu$ in the equations for the coefficients $A_{1/3}$ and $B_{1/3}$, which can be easily obtained from the relation (1.6) and the one analogous to it. As a result of the computations we get

$$A_{1/3}^3 = -\frac{4}{3}\frac{(a\nu - b\lambda)^3}{(a^2 + b^2)^2(\nu^2 + \lambda^2)}, \qquad B_{1/3}^3 = -\frac{4}{3}\frac{(a\lambda + b\nu)^3}{(a^2 + b^2)^2(\nu^2 + \lambda^2)} \quad (1.22)$$

Since there is only one pair of real values of the coefficients $A_{1/3}$ and $B_{1/3}$, there will be only one real periodic solution of Eqs. (1.18) which can be expanded in a power of series of $\mu^{1/3}$. To obtain the other coefficients of this series we use the method of successive integration of the equations for $x_{n/3}(t)$

Skipping the derivations, we get as a final result

$$x(t) = \mu^{1/3} x_{1/3}(t) + \mu^2 x_2(t) + \mu^{11/3} x_{11/3}(t) + \ldots \quad (1.23)$$

The remaining intermediate terms of the series are equal to zero. The coefficients $x_{1/3}(t)$ and $x_2(t)$ have the following values

$$x_{1/3}(t) = A_{1/3}\cos t + B_{1/3}\sin t \tag{1.24}$$

$$x_2(t) = \frac{1}{96}\frac{A_{1/3}^2 + B_{1/3}^2}{a^2+b^2}[(eA_{1/3} - gB_{1/3})\cos t + (gA_{1/3} + eB_{1/3})\sin t] + \\ + \frac{1}{32}[(aP+bQ)\cos 3t + (bP - aQ)\sin 3t] \tag{1.25}$$

Here we have introduced the notation

$$e = a(a^2 - 7b^2), \quad g = 3b(5a^2 - 3b^2)$$

$$P = A_{1/3}(A_{1/3}^2 - 3B_{1/3}^2), \qquad Q = B_{1/3}(B_{1/3}^2 - 3A_{1/3}^2)$$

The coefficients $x_{11/3}(t)$ contains the first, the third and the fifth harmonics. For Duffing's equation with $b = 0$ and $\lambda = 0$ we get

$$A_{1/3}^3 = -\frac{4}{3}\frac{\nu}{a}, \quad B_{1/3} = 0, \quad x_2(t) = -\frac{\nu}{32}\left(\frac{1}{3}\cos t + \cos 3t\right)$$

2. There are no self oscillations in the generating functions ($m = 0$). In the given case, the general solution of the generating function is aperiodic

$$x_0(t) = \varphi(t) + A_0 + B_0 t \tag{2.1}$$

Let us take the same initial conditions as in the first case, that is in the form (1.2) for the system (0.1) under investigation. The solution of the system (0.1) for $m = 0$ has the form

$$x(t) = \varphi(t) + A_0 + \beta + (B_0 + \gamma)t + \sum_{n=1}^{\infty}\left[C_n(t) + \frac{\partial C_n(t)}{\partial A_0}\beta + \frac{\partial C_n(t)}{\partial B_0}\gamma + \cdots\right]\mu^n \tag{2.2}$$

The functions $C_n(t)$ and their first derivatives with respect to t are determined by means of Formulas

$$C_n(t) = \int_0^t H_n(t')(t - t')\,dt', \qquad C_n^{\cdot}(t) = \int_0^t H_n(t')\,dt' \tag{2.3}$$

From the conditions of periodicity of the solution $x(t)$ and its first derivatives with respect to t we have

$$2\pi(B_0 + \gamma) + \sum_{n=1}^{\infty}\left[C_n(2\pi) + \frac{\partial C_n}{\partial A_0}\beta + \frac{\partial C_n}{\partial B_0}\gamma + \frac{1}{2}\frac{\partial^2 C_n}{\partial A_0^2}\beta^2 + \cdots\right]\mu^n = 0$$
$$\sum_{n=1}^{\infty}\left[C_n^{\cdot}(2\pi) + \frac{\partial C_n^{\cdot}}{\partial A_0}\beta + \frac{\partial C_n^{\cdot}}{\partial B_0}\gamma + \frac{1}{2}\frac{\partial^2 C_n^{\cdot}}{\partial A_0^2}\beta^2 + \cdots\right]\mu^{n-1} = 0 \tag{2.4}$$

Substituting in these equalities $\beta = \gamma = \mu = 0$, we get the amplitude Eqs.

$$2\pi B_0 = 0, \qquad C_1^{\cdot}(2\pi, A_0, B_0) = 0 \tag{2.5}$$

which reduces to a single equation with respect to A_0.

In the case of simple roots of the amplitude equation, we get an infinite system of pairs of linear equations for the coefficients A_n and B_n. The equations for A_1 and B_1 are:

$$2\pi B_1 + C_1 = 0, \qquad \frac{\partial C_1^{\cdot}}{\partial A_0}A_1 + \frac{\partial C_1^{\cdot}}{\partial B_0}B_1 + C_2^{\cdot} = 0 \tag{2.6}$$

The equations for the coefficients A_2 and B_2 are

$$2\pi B_2 + \frac{\partial C_1}{\partial A_0}A_1 + \frac{\partial C_1}{\partial B_0}B_1 + C_2 = 0$$

$$\frac{\partial C_1^{\cdot}}{\partial A_0} A_2 + \frac{\partial C_1^{\cdot}}{\partial B_0} B_2 + \frac{1}{2} \frac{\partial^2 C_1^{\cdot}}{\partial A_0^2} A_1^2 + \frac{\partial^2 C_1^{\cdot}}{\partial A_0 \partial B_0} A_1 B_1 + \frac{1}{2} \frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} B_1^2 + $$
$$+ \frac{\partial C_2^{\cdot}}{\partial A_0} A_1 + \frac{\partial C_2^{\cdot}}{\partial B_0} B_1 + C_3^{\cdot} = 0 \tag{2.7}$$

This system can always be solved since $\partial C_1^{\cdot} / \partial A_0 \neq 0$.

In the case of repeated roots of the amplitude equation let us express from the first equation in (2.4) the parameter γ in the function of $A_0 + \beta$ and μ:

$$2\pi\gamma = P_1\mu + \frac{\partial P_1}{\partial A_0}\beta\mu + P_2\mu^2 + \frac{1}{2}\frac{\partial^2 P_1}{\partial A_0^2}\beta^2\mu + \frac{\partial P_2}{\partial A_0}\beta\mu^2 + P_3\mu^3 + \cdots \tag{2.8}$$

The coefficients P_n are computed from Expressions

$$P_1 = -C_1, \qquad P_2 = \frac{1}{2\pi}\frac{\partial C_1}{\partial B_0} C_1 - C_2 \tag{2.9}$$

$$P_3 = -\frac{1}{4\pi^2}\left[\left(\frac{\partial C_1}{\partial B_0}\right)^2 + \frac{1}{2}\frac{\partial^2 C_1}{\partial B_0^2} C_1\right] C_1 + \frac{1}{2\pi}\left(\frac{\partial C_1}{\partial B_0} C_2 + \frac{\partial C_2}{\partial B_0} C_1\right) - C_3$$

Let us introduce the expression for γ in the second equation in (2.4). We get relations of the form

$$\sum_{n=1}^{\infty} Q_n (A_0 + \beta)\mu^{n-1} = 0$$

whereupon $Q_1 = C_1^{\cdot}(2\pi, A_0, B_0) = 0$. In the developed form the expression for the parameter β is

$$\Phi^* (\beta, \mu) = \frac{\partial C_1^{\cdot}}{\partial A_0}\beta + Q_2\mu + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\beta^2 + \frac{\partial Q_2}{\partial A_0}\beta\mu + Q_3\mu^2 + $$
$$+ \frac{1}{6}\frac{\partial^3 C_1^{\cdot}}{\partial A_0^3}\beta^3 + \frac{1}{2}\frac{\partial^2 Q_2}{\partial A_0^2}\beta^2\mu + \frac{\partial Q_3}{\partial A_0}\beta\mu^2 + Q_4\mu^3 + \cdots = 0 \tag{2.10}$$

Computing the coefficients Q_n, we get

$$Q_2 = -\frac{1}{2\pi}\frac{\partial C_1^{\cdot}}{\partial A_0} C_1 + C_2^{\cdot}$$

$$Q_3 = \frac{1}{4\pi^2}\left(\frac{\partial C_1^{\cdot}}{\partial B_0}\frac{\partial C_1}{\partial B_0} + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} C_1\right) C_1 - \frac{1}{2\pi}\left(\frac{\partial C_1^{\cdot}}{\partial B_0} C_2 + \frac{\partial C_2^{\cdot}}{\partial B_0} C_1\right) + C_3^{\cdot}$$

$$Q_4 = \frac{1}{2\pi}\frac{\partial C_1^{\cdot}}{\partial B_0} P_3 + \frac{1}{2\pi}\left(\frac{\partial C_2^{\cdot}}{\partial B_0} - \frac{1}{2\pi}\frac{\partial^2 C_1^{\cdot}}{\partial B_0^2} C_1\right) P_2 - \frac{1}{48\pi^3}\frac{\partial^3 C_1^{\cdot}}{\partial B_0^3} C_1^3 + $$
$$+ \frac{1}{8\pi^2}\frac{\partial^2 C_2^{\cdot}}{\partial B_0^2} C_1^2 - \frac{1}{2\pi}\frac{\partial C_3^{\cdot}}{\partial B_0} C_1 + C_4^{\cdot} \tag{2.11}$$

For $\mu = 0$ we get

$$\Phi^* (\beta, 0) = \frac{\partial C_1^{\cdot}}{\partial A_0}\beta + \frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}\beta^2 + \frac{1}{6}\frac{\partial^3 C_1^{\cdot}}{\partial A_0^3}\beta^3 + \cdots \tag{2.12}$$

Thus, in the given particular case, the problem of the determination of the parameters β and γ, is reduced to the solution of one equation for the parameter β, as was done in the general case [3]. The analysis of the solution of this equation in the case of roots of the amplitude equation repeated twice and three times is given in [4].

The form of the expansion of the periodic solutions of the Eq. (0.1) is determined in the form of expansions of the parameters β and γ. Let us assume for instance that the solution is expanded in power of $\mu^{1/2}$

$$x(t) = x_0(t) + \mu^{1/2} x_{1/2}(t) + \mu x_1(t) + \ldots \tag{2.13}$$

The coefficients of this expansion for $m = 0$ can be determined by means of the Formulas
$$x_{1/2}(t) = A_{1/2}, \quad x_1(t) = A_1 + B_1 t + C_1(t), \quad x_{3/2}(t) = A_{3/2} + B_{3/2}t + A_{1/2}\frac{\partial C_1(t)}{\partial A_0}$$
$$x_2(t) = A_2 + B_2 t + A_1\frac{\partial C_1(t)}{\partial A_0} + B_1\frac{\partial C_1(t)}{\partial B_0} + \frac{1}{2}A_{1/2}^2\frac{\partial^2 C_1(t)}{\partial A_0^2} + C_2(t) \qquad (2.14)$$
etc. In the given case it is convenient to find also these coefficients directly by integrating the differential equation determining them.

As an example let us consider the system
$$x'' = \mu\,[\cos t + f_0(x) + x' f_1(x) + x'^2 f_2(x)] \qquad (2.15)$$
where the functions $f_n(x)$ have derivatives of any order.

Forming the amplitude equation for the given example, we get
$$C_1^{\cdot}(2\pi) = 2\pi f_0(A_0) = 0 \qquad (2.16)$$
Let us take some real root A_0 of this equation. We shall consider two cases:

1) Case of a single root $f_0'(A_0) \neq 0$. Integrating in succession the equations for $x_n(t)$, we find periodic solutions of the Eq. (2.15) with an accuracy up to μ^2, inclusively
$$x(t) = A_0 - \mu\cos t + \mu^2\,[E_{21} + f_0'(A_0)\cos t - f_1(A_0)\sin t] + \ldots \qquad (2.17)$$
where
$$E_{21} = -{}^1/_2\,[{}^1/_2 f_0''(A_0) + f_2(x)]\,[f_0'(A_0)]^{-1} \qquad (2.18)$$
2) Case of a double root $f_0'(A_0) = 0$, but $f_0''(A_0) \neq 0$. The computations show that in that case the quantity $Q_2 = 0$ and the equation for the coefficient A_1
$$\frac{1}{2}\frac{\partial^2 C_1^{\cdot}}{\partial A_0^2}A_1^2 + \frac{\partial Q_2}{\partial A_0}A_1 + Q_3 = 0$$
has single roots. Thus the parameters β and γ are expanded according to integer powers of μ. We get
$$x_1(t) = -\cos t + E_{12}, \qquad x_2(t) = -f_1(A_0)\sin t + E_{22}$$
From the periodicity conditions for the function $x_3(t)$ we get
$$E_{12} = \pm\sqrt{-(\alpha + {}^1/_2)}, \qquad \alpha = f_2(A_0)\,/\,f_0''(A_0) \qquad (2.19)$$
Consequently, for E_{12} to be real it is necessary that
$$f_0''(A_0) + 2f_2(A_0) \leqslant 0 \qquad (2.20)$$
From the periodicity conditions of $x_4(t)$ we get
$$E_{12}\,[E_{22}f_0''(A_0) + {}^1/_6 E_{12}^2 f_0'''(A_0) + {}^1/_4 f_0'''(A_0) + {}^1/_2 f_2'(A_0)] = 0 \qquad (2.21)$$
It follows that if $\alpha \neq -{}^1/_2$, we have
$$E_{22} = -{}^1/_2\,[{}^1/_3(1-\alpha)f_0'''(A_0) + f_2'(A_0)]\,[f_0''(A_0)]^{-1} \qquad (2.22)$$
Thus the periodic solution for a root repeated twice is, with the same accuracy
$$x(t) = A_0 + \mu\,[\pm\sqrt{-(\alpha + {}^1/_2)} - \cos t] + \mu^2\,[E_{22} - f_1(A_0)\sin t] + \cdots \qquad (2.23)$$

For $\alpha = -{}^1/_2$ the constant integration of E_{12} becomes zero. The quantity E_{22} is not determined from the condition (2.21). A necessary additional investigation is necessary since the form of the expansion of $x(t)$ can change for $\alpha = -{}^1/_2$.

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